FINITE AND TORSION KK-THEORIES

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Abstract. We develop a finite KK^G-theory of $C*$-algebras following Arlettaz-H.Inassaridze's approach to finite algebraic K-theory [1] . The Browder-Karoubi-Lambre's theorem on the orders of the elements for finite algebraic K-theory [,] is extended to finite KK^G-theory. A new bivariant theory, called torsion KK-theory is defined as the direct limit of finite KK-theories. Such bivariant K-theory has almost all KK^G-theory properties and one has the following exact sequence

$$\cdots \;\to\; KK_n^G(A,B) \;\to\; KK_n^G(A,B;\mathbb{Q}) \;\to\; KK_n^G(A,B;\mathbb{T}) \;\to\; \cdots$$

relating KK-theory, rational bivariant K-theory and torsion KK-theory. For a given homology theory on the category of separable GC^* -algebras finite, rational and torsion homology theories are introduced and investigated. In particular, we formulate finite, torsion and rational versions of Baum-Connes Conjecture. The later is equivalent to the investigation of rational and q-finite analogues for Baum-Connes Conjecture for all prime q.

Introduction

In this paper we provide a new bivariant theory, which will be called torsion equivariant KK^G-theory. That is closely connected with the usual and rational versions of KK^G-theories. By definition torsion KK^G-theory is a direct limit of KK^G-theory with coefficients in Z_q (q-finite KK^G-theory in our terminology), where q runs over all natural numbers ≥ 2. This new bivariant homology theory has all the properties of KK^G-theory except of the existence of the identity morphism. We arrive to the following principle: some of problems that arise in usual KK^G-theory may be reduced to suitable problems in rational, finite and torsion KK^G-theories. Namely, it will be shown that Baum-Connes conjecture has analogues in finite, torsion and rational KK-theories and Baum-Connes assembly map is an isomorphism if and only if its rational and finite assembly maps are isomorphisms for all prime q (Theorem 3.5).

As a technical tool, we mainly work with homology theories on the category of C^*-algebras with action of a fix locally compact group G. In sections 1 and 2 for a given homology theory H torsion and q-finite homology theories $H^{(q)}$ are constructed and their properties are investigated. Much of these properties are known for experts in some concrete form, but we could not find suitable references for our purposes. They are redefined and reinvestigated here. Furthermore, in section 2 we define and investigate a new homology theory, so called torsion homology theory. Especially, we make accent on the following twosided long exact sequence of abelian

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groups

$$\cdots \to H^{\mathbb{T}}_{n+1}(A) \to H_n(A) \xrightarrow{r} H_n(A)\otimes \mathbb{Q} \to H^{\mathbb{T}}_n(A) \to H_{n-1}(A) \xrightarrow{r} \cdots$$

for any GC^*-algebra A which is used concretely for bivariant KK-theories in the sequel section. In particular, based on results of these sections we list properties of torsion and finite bivariant KK^G-theories. Besides, there exists a long exact sequence, which is similar to the above long exact sequence:

$$\begin{aligned}(0.1)\quad \cdots \to KK^G_{n+1}(A,B;\mathbb{Q}) \to KK^G_{n+1}(A,B;\mathbb{Q}/\mathbb{Z}) \to \\ \to KK^G_n(A,B) \xrightarrow{Rat_n} KK^G_n(A,B;\mathbb{Q}) \to KK^G_n(A,B;\mathbb{Q}/\mathbb{Z})) \to \cdots\end{aligned}$$

The similar result for K-theory of bornological algebras one can find in [5].

The rational bivariant KK-theory and the torsion bivariant KK-theory have all the properties of usual bivariant KK-theory. The only difference is that the torsion case hasn't unital morphisms. Note that rational bivariant KK-theory used in this paper differs from the similar one defined in [3].

In the next section 3 we study torsion and q-finite KK-theories, where the finite KK-theory is redefined following Arlettaz-H.Inassaridze's approach to finite algebraic K-theory [1]. Sections 4 and 5 are devoted to the proof of the following Browder-Karoubi-Lambre' theorem for finite KK-theory (see Theorem 5.5):

Let A and B be, respectively, separable and σ-unital C^*-algebras, real or complex; and G be a metrizable compact group. Then, for all integer n,

(1) $q \cdot KK^G_n(A,B;Z/q) = 0,$ if $q-2$ is not divided by 4;
(2) $2q \cdot KK^G_n(A,B;Z/q) = 0,$ if 4 divides $q-2$.

It is clear that this result holds for non-unital rings too. For finite algebraic K-theory this theorem for $n = 1$ was proved algebraically by Karoubi and Lambre [], and for $n > 1$ by Browder [2].

The key idea to carry out this problem is its reduction to the algebraic K-theory case. This is realized by two steps. First we calculate finite topological K-theory of C^*-algebras and additive C^*-categories by finite algebraic K-theory of rings. Then generalizing the main result of [8], finite bivariant KK^G-theory is calculated by finite topological K-theory of the additive C^*-category of Fredholm modules. When G is a locally compact group, it is more complicated to get the similar result for finite G-equivariant bivariant KK-theory and we intend to investigate this problem in a forthcoming paper.

1. On Finite homology theory

In this section we analyze some properties of homology theory with coefficients in $\mathbb{Z}_q$ which is said to be *q-finite homology*. There exist some different ways to construct for a given homology theory on C^*-algebras a corresponding q-finite homology theory; we choose one of them, suitable for our purposes.

Let S^1 be the unit cycle in the plane of complex numbers with module one. The map

$$\tilde{q} : S^1 \to S^1, \qquad x \mapsto x^q,$$

$q \geq 2$, $q \in \mathbb{N}$, is called *standard q-th power map.* Since $1 \in S^1$ is invariant relative to the map $\tilde{q}$, it can be considered as a map of pointed spaces

$$\tilde{q} : S^1_* \to S^1_*, \qquad x \mapsto x^q,$$

where $* = 1$. These are basic q-th power maps in algebra and topology.

Let $C_0(S^1)$ be a C^*-algebra of continuous complex (or real) functions on the unit cycle S^1 in the plane of complex numbers with module one vanishing at 1. Then the map

$$\tilde{q} : S^1 \to S^1, \qquad x \mapsto x^q,$$

$q \geq 2$, $q \in \mathbb{N}$, induces a $*$-homomorphism

$$\hat{q} : C_0(S^1) \to C_0(S^1), \qquad f(s) \mapsto f(s^q).$$

Denote C^*-algebra C_q as cone of the homomorphism $\hat{q}$:

$$C_q = \{(x, f) \in C_0(S^1) \oplus C_0(S^1) \otimes C[0; 1) \mid \ \hat{q}(x) = f(0)\},$$

The following lemma is one of the main property of the degree map. The idea of the proof is taken from [13].

Lemma 1.1. *Let $p_q : C_{pq} \to C_q$ be a natural map induced by a commutative*

$$\begin{tikzcd}
C_0(S^1) \arrow[r, "p"] \arrow[d, "pq"'] & C_0(S^1) \arrow[d, "q"] \\
C_0(S^1) \arrow[r, "="] & C_0(S^1)
\end{tikzcd}$$

diagram. Then there is a natural homomorphism $\nu_{p,q} : C_{p_q} \to C_p$, which is a homotopy equivalence.

Proof. A homomorphism $\nu_{p,q}$ is induced by the commutative diagram

$$\begin{tikzcd}
C_{pq} \arrow[r, "p_q"] \arrow[d] & C_q \arrow[d] \\
C_0(S^1) \arrow[r, "p"] & C_0(S^1).
\end{tikzcd}$$

Choose a homotopy $H : [0, 1]^2 \times [0, 1] \to [0, 1]^2$ relative to $L = [0, 1] \times \{1\} \cup \{0, 1\} \times [0, 1]$ such that $H_0 = id$ and H_1 is the retraction of $[0, 1]^2$ on L. Then, a homotopy inverse to $\nu_{p,q}$ is given by

$$\chi : C_p \to C_{p_q}, \quad \chi(a, b) = (a, b, \tilde{b} \cdot H_1)$$

For $b \in C_0(S^1)[0, 1)$ define $\tilde{b} : L \to C_0(S^1)$ by $\tilde{b}(s, 1) = (1, t) = 0$ and $\tilde{b}(0, t) = b(t^q), \ \ t, s \in [0, 1]$. We have $\nu_{p,q} \cdot \chi = C_p$.

There is a homotopy between $id_{C_{p_q}}$ and $\chi \cdot \nu_{p,q}$ which is given by a map

$$G_t(a, b, c) = (a, b, c_t)$$

where $c_t(r, s) = c(H_t(r, s))$. $\square$

Lemma 1.2. (1) *Let A be a C^*-algebra. Then the commutative diagram*

$$\begin{tikzcd} A \otimes C_q \arrow[r] \arrow[d] & A \otimes C_0(S^1)^{[0,1)} \arrow[d] \\ A \otimes C_0(S^1) \arrow[r, "id_A \otimes \hat{q}"] & A \otimes C_0(S^1) \end{tikzcd}$$

is a pullback diagram. In particular, $A \otimes C_q \simeq C_{id_A \otimes \hat{q}}$.

(2) *Let the diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

be a pullback diagram and (X, x) pointed compact space. Then the induced diagram

$$\begin{tikzcd} A^{(X,x)} \arrow[r] \arrow[d] & B^{(X,x)} \arrow[d] \\ C^{(X,x)} \arrow[r] & D^{(X,x)} \end{tikzcd}$$

is a pullback diagram.

Proof. (1). Let the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & A \otimes C_0(S^1)^{[0,1)} \arrow[d] \\ A \otimes C_0(S^1) \arrow[r, "id_A \otimes \hat{q}"] & A \otimes C_0(S^1) \end{tikzcd}$$

be a pullback diagram. Then P contains the couple of functions $(f(s), g(s,t))$, such that $f(s^q) = g(s, 0)$, $f(0) = 0$, $g(0, t) = 0$; $s \in S^1$ $t \in [0, 1)$. Therefore the pair $(f(s), g(s,t))$ defines a continuous function on the cone σ_q of the degree map $S^1 \xrightarrow{q} S^1$ with values in A . So, there is a homomorphism $P \to A^{\sigma_q} \simeq A \otimes C_q$ (which is a morphism of suitable diagrams). Thus the diagram is pullback and as a consequence we get the isomorphism $A \otimes C_q \simeq C_{id_A \otimes \hat{q}}$.

(2) is trivial. □

Recall that a family of functors $H = \{H_n\}_{n \in \mathbb{Z}}$ on the category of (separable or σ-unital) GC^*-algebras (real or complex) [9] is said to be homology theory (cf. [3]): if

(1) H_n is a homotopy invariant functor for any $n \in Z$
(2) for any $*$-homomorphism (G-equivariant) of σ-unital algebras $f : A \to B$ there exists a natural twosided long exact sequence of abelian groups:

$$\cdots \to H_{n+1}(B) \to H_n(C_f) \to H_n(A) \to H_n(B) \to H_{n-1}(C_f) \to \cdots$$

where C_f is the cone of f.

Definition 1.3. Under the q-finite homology of a homology H, $q \geq 2$, we mean a family of functors $H^{(q)} = \{H_n^{(q)}\}_{n\in\mathbb{Z}}$, where

$$H_n^{(q)} = H_{n-2}(- \otimes C_q).$$

Below we list main properties of the q-finite homology.

Proposition 1.4. *Let H be a homology theory on the category of C^*-algebras. Then*

(1) *$H^{(q)}$ is a homology theory;*
(2) *there is a twosided long exact sequence of abelian groups*

$$\cdots \to H_{n+1}^{(q)}(A) \to H_n(A) \xrightarrow{q\,\cdot} H_n(A) \to H_n^{(q)}(A) \to H_{n-1}(A) \xrightarrow{q\,\cdot} \cdots$$

(3) *there is a twosided long exact sequence of abelian groups*

$$\cdots \to H_{n+1}^{(p)}(A) \to H_n(A)^{(q)} \xrightarrow{\dot{p}} H_n^{(pq)}(A) \xrightarrow{\acute{q}} H_n^{(p)}(A) \to H_{n-1}^{(q)}(A) \xrightarrow{\dot{p}} \cdots$$

(4) *if homology H has an associative product*

$$H_n(A) \otimes H_m(B) \to H_{n+m}(A \otimes B)$$

then there is an associative product

$$H_n^{(p)}(A) \otimes H_q^{(q)}(B) \to H_{n+m-2}^{(pq)}(A \otimes B).$$

Proof. The first and the second parts are immediate consequences of Lemma 1.2 and Definition 1.3. The third is a trivial consequence of the Puppe's exact sequence for the homomorphism $p_q : A \otimes C_{pq} \to A \otimes C_q$, Lemma 1.4 and Lemma 1.2. Finally we have

$$\begin{aligned}(1.1)\quad H_n^{(p)}(A) \otimes H_m^{(q)}(B) = H_{n-2}(A \otimes C_p) \otimes H_{m-2}^{(q)}(B \otimes C_q) \to \\ \to H_{n+m-4}(A \otimes B \otimes C_p \otimes C_q) \to H_{n+m-4}(A \otimes B \otimes C_{pq}) \to H_{n+m-2}^{(pq)}(A \otimes B)\end{aligned}$$

where the product $C_p \otimes C_q \to C_{pq}$ is defined as follows. There are natural homomorphisms $\breve{q} : C_p \to C_{pq}$, induced by the commutative diagram

$$\begin{tikzcd} C_p \arrow[r] \arrow[d, "\breve{q}"'] & C_0(S^1) \arrow[r, "p"] \arrow[d, "="] & C_0(S^1) \arrow[d, "q"] \\ C_{pq} \arrow[r] & C_0(S^1) \arrow[r, "pq"] & C_0(S^1), \end{tikzcd}$$

and similarly $\breve{p} : C_q \to C_{pq}$. Since all algebras are nuclear (in C^*-algebraic sense), these homomorphisms yield a homomorphism (product) $C_p \otimes C_q \to C_{pq}$ which is associative in the obvious sense. □

2. On Torsion Homology theory

Now we define a new homology theory using the family of q-finite homology theories, $q \geq 2$. Consider the ordered set $\mathbb{N}_{(2)} = \{q \in \mathbb{N} \mid q \geq 2\}$, where $q \leq q'$ iff q divides q'.

Note that if $q' = qs$, then there is a natural transformation of functors

$$\tau_n^{(qq')} : H_n^{(q)} \to H_n^{(q')}$$

induced by the homomorphism q_s,

$$\begin{tikzcd} C_q \arrow[r] \arrow[d, "q_s"'] & C_0(S^1) \arrow[r, "q"] \arrow[d, "="] & C_0(S^1) \arrow[d, "s"] \\ C_{q'} \arrow[r] & C_0(S^1) \arrow[r, "q'"] & C_0(S^1), \end{tikzcd}$$

where $\tau_n^{(qq')}(A) : H_n^{(q)}(A) \to H_n^{(q')}(A)$ denotes the homomorphism $H_n(id_A \otimes q_s)$. Therefore one has an inductive system of abelian groups

$$\{H_n^{(q)}(A), \tau_n^{(qq')}(A)\}_{q\in\mathbb{N}_{(2)}}$$

for any GC^*-algebra A.

Proposition 2.1. *Let H be a homology theory and $H^{\mathbb{T}}$ be a family of functors defined by the equality*

$$H_n^{\mathbb{T}}(A) = \varinjlim_q \ H_n^{(q)}(A), \quad n \in \mathbb{Z}, \quad q \geq 2,$$

for any C^*-algebra A. Then

(1) *$H^{\mathbb{T}}$ is a homology theory H on the category GC^*-algebras;*
(2) *There is a twosided long exact sequence of abelian groups*

$$\cdots \to H_{n+1}^{\mathbb{T}}(A) \to H_n(A) \xrightarrow{r} H_n(A) \otimes \mathbb{Q} \to H_n^{\mathbb{T}}(A) \to H_{n-1}(A) \xrightarrow{r} \cdots$$

for any GC^*-algebra A.

(3) *There is a twosided long exact sequence of abelian groups*

$$\cdots \to H_{n+1}^{\mathbb{T}}(A) \xrightarrow{\hat{q}} H_{n+1}^{(q)}(A) \to H_n^{\mathbb{T}}(A) \xrightarrow{\hat{p}} H_n^{\mathbb{T}}(A) \xrightarrow{\hat{q}} H_n^{(q)}(A) \to \cdots$$

for any GC^*-algebra A.

Proof. According to Proposition 1.4 (1) the first part is an easy consequence of the fact that the direct limit preserves homotopy and excision properties. For the second part consider the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}^{(q)}(A) \arrow[r] \arrow[d] & H_n(A) \arrow[r, "q"] \arrow[d, equal] & H_n(A) \arrow[r] \arrow[d, "\frac{q'}{q}"] & H_n^{(q)}(A) \arrow[r] \arrow[d, "\tau_n^{(qq')}(A)"] & \cdots \\ \cdots \arrow[r] & H_{n+1}^{(q')}(A) \arrow[r] & H_n(A) \arrow[r, "q'"] & H_n(A) \arrow[r] & H_n^{(q')}(A) \arrow[r] & \cdots, \end{tikzcd}$$

where rows are long twosided exact sequences. By taking the direct limit of these long exact sequences, one gets the following long twosided exact sequence

$$\cdots \longrightarrow H_n(A) \xrightarrow{\hat{q}} \varinjlim_q H_n(A) \longrightarrow H_n^{\mathbb{T}}(A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

It is easy to check that the inductive system $\{H_n(A), \frac{q'}{q}\}$ is isomorphic to the inductive system $\{H_n(A) \otimes \mathbb{Z}^{\{q\}}, \frac{q'}{q}\}$, where $\mathbb{Z}^{\{q\}} = \mathbb{Z}$ for all q. Then

$$\varinjlim_q H_n(A) \simeq H_n(A) \otimes \varinjlim_q \mathbb{Z}^{\{q\}} \simeq H_n(A) \otimes \mathbb{Q},$$

since one has the isomorphism $\varinjlim_q \mathbb{Z}^{\{q\}} \simeq \mathbb{Q}$ defined by the map $(q, r) \mapsto \frac{r}{q}$.

For (3) consider the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{(q)}_{n+1}(A) \arrow[r] \arrow[d] & H^{(p)}_{n}(A) \arrow[r, "p"] \arrow[d] & H^{(pq)}_{n}(A) \arrow[r] \arrow[d, "\frac{q'}{q}"] & H^{(q)}_{n}(A) \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r] & H^{(q)}_{n+1}(A) \arrow[r] & H^{(p')}_{n}(A) \arrow[r, "q"] & H^{(pq)}_{n}(A) \arrow[r] & H^{(q)}_{n}(A) \arrow[r] & \cdots,
\end{tikzcd}$$

where rows are long twosided exact sequences given in Proposition 1.4 (3). The direct limit of these long exact sequences with respect to p yields the required long twosided exact sequence. □

The homology theory $H^{\mathbb{T}}$ is said to be the torsion homology of the homology H.

Corollary 2.2. *Let $\tau : H \to \tilde{H}$ be a natural transformation of homology theories. Then τ induces natural transformations $\tau^{(q)} : H^{(q)} \to \tilde{H}^{(q)}$, $\tau^{\mathbb{T}} : H^{\mathbb{T}} \to \tilde{H}^{\mathbb{T}}$ and $\tau_{\mathbb{Q}} : H \otimes \mathbb{Q} \to \tilde{H} \otimes \mathbb{Q}$. Furthermore the following conditions are equivalent.*

(1) *$\tau(A)$ is an isomorphism for a C^*-Algebra A.*
(2) *$\tau^{\mathbb{T}}(A)$ and $\tau_{\mathbb{Q}}(A)$ is an isomorphism for a C^*-Algebra A.*
(3) *$\tau^{(q)}(A)$ for all prims q and $\tau_{\mathbb{Q}}(A)$ is an isomorphism for a C^*-Algebra A.*

Proof. (1) $\cong$ (2) is consequence of the Five Lemma and following commutative diagram of twosided long exact sequence:

$$\begin{tikzcd}
\cdot\cdot \arrow[r] & H^{\mathbb{T}}_{n+1}(A) \arrow[r] \arrow[d, "\tau^{\mathbb{T}}(A)"] & H_n(A) \arrow[r] \arrow[d, "\tau(A)"] & H_n(A) \otimes \mathbb{Q} \arrow[r] \arrow[d, "\tau_{\mathbb{Q}}(A)"] & H^{\mathbb{T}}_{n}(A) \arrow[r] \arrow[d, "\tau^{\mathbb{T}}(A)"] & \cdot\cdot \\
\cdot\cdot \arrow[r] & \tilde{H}^{\mathbb{T}}_{n+1}(A) \arrow[r] & \tilde{H}_n(A) \arrow[r] & \tilde{H}_n(A) \otimes \mathbb{Q} \arrow[r] & \tilde{H}^{\mathbb{T}}_{n}(A) \arrow[r] & \cdot\cdot
\end{tikzcd}$$

(2) $\cong$ (3) is a consequence of the Five Lemma and following commutative diagrams of the twosided long exact sequence:

$$\begin{tikzcd}
\cdot\cdot \arrow[r] & H^{(q)}_{n+1}(A) \arrow[r] \arrow[d, "\tau^{(q)}"] & H^{\mathbb{T}}_{n}(A) \arrow[r] \arrow[d, "\tau^{\mathbb{T}}"] & H^{\mathbb{T}}_{n}(A) \arrow[r] \arrow[d, "\tau^{\mathbb{T}}"] & H^{(q)}_{n}(A) \arrow[r] \arrow[d, "\tau^{(q)}"] & \cdot\cdot \\
\cdot\cdot \arrow[r] & \tilde{H}^{(q)}_{n+1}(A) \arrow[r] & \tilde{H}^{\mathbb{T}}_{n}(A) \arrow[r] & \tilde{H}^{\mathbb{T}}_{n}(A) \arrow[r] & \tilde{H}^{(q)}_{n}(A) \arrow[r] & \cdot\cdot
\end{tikzcd}$$

and

$$\begin{tikzcd}
\cdot\cdot \arrow[r] & H^{(q)}_{n+1}(A) \arrow[r] \arrow[d, "\tau^{(q)}"] & H^{(p)}_{n}(A) \arrow[r] \arrow[d, "\tau^{(p)}"] & H^{(pq)}_{n}(A) \arrow[r] \arrow[d, "\tau^{(pq)}"] & H^{(q)}_{n}(A) \arrow[r] \arrow[d, "\tau^{(q)}"] & \cdots \\
\cdot\cdot \arrow[r] & \tilde{H}^{(q)}_{n+1}(A) \arrow[r] & \tilde{H}^{(p)}_{n}(A) \arrow[r] & \tilde{H}^{(pq)}_{n}(A) \arrow[r] & \tilde{H}^{(q)}_{n}(A) \arrow[r] & \cdot\cdot
\end{tikzcd}$$

□

3. Applications to KK-theory

3.1. **Torsion and finite KK-theories.** By considering $KK^G(A, -)$ as a homology theory and according to section 1 we define finite, torsion and rational KK^G-theories for all integer n as follows.

Definition 3.1.

$$KK_n^G(A,B;\mathbb{Z}_q) = KK_{n-2}^G(A, B\otimes C_q). \tag{3.1}$$

Definition 3.2.

$$KK_n^G(A,B,T) = \varinjlim_q KK_n^G(A,B;\mathbb{Z}_q)$$

.

Definition 3.3.

$$KK_n^G(A,B;Q) = KK_n^G(A,B)\otimes Q.$$

Our definitions of finite and rational KK^G differ from the exisiting definitions of finite and rational KK-theories ([3], 23.15.6-7). In effect, here we compare the two versions of the definitions of finite and rational KK- theories.

1. Let N be the smallest class of separable C-algebras with the following properties:

(N1) N contains field complex numbers;

(N2) N is closed under countable inductive limits;

(N3) if $0\to A\to D\to B\to 0$

is an exact sequence, and two of them are in N, then so is the third;

(N4) N is closed under KK-equivalence.

Let D be a C^*-algebra in N with $K_0(D)=Z_p, K_1(D)=0$.

Define

$$KK_n(A;B;Zp) = KK_n(A;B\otimes D).$$

As noted in ([3]), so defined KKS-groups are independent of the choice of D. Bellow we show that the above definition is equivalent to our definition. One has $K_0(C_m)=Z_m$ and $K_1(C_m)=0$. This is an easy consequence of the Bott periodicity theorem and the two-sided long exact sequence

$$....\to K_2(C_0(S^1))\to K_2(C_0(S^1))\to K_2(C_m)\to K_1(C_0(S^1))\to ...,$$

since $K_1(C_0(S^1))=0$.

Therefore our definition of finite KK-theory agrees to its definition in the sense of [1] taking into account the following isomorphism induced by the Bott periodicity theorem:

$$KK_n(A;B\otimes C_m(S^1))\simeq KK_{n-2}(A;B\otimes C_m(S_1)).$$

2. The rational KK-theory is defined in ([1], 23.15.6) by the following manner. Let D be a C^*-algebra in N with $K_0(D)=Q, K_1(D)=0$. Define

$$KK_n(A;B;Q) = KK_n(A;B\otimes D).$$

In general $KK_n(A;B;Q)\neq KK_n(A;B)\otimes Q$ ([3], 23.15.6). For example,

$$KK(D;C;Q)=Q \quad \text{and} \quad KK(D;C)\otimes Q = 0.$$

This means that our rational KK-theory differs from that of [3].

According to results of the previous section one has the following properties of q-finite and torsion KK-theories.

(1) The groups $KK^G(A,B;\mathbb{Z}_q)$ have Bott periodicity property and satisfy the excision property relative to both arguments.

(2) there is a natural twosided exact sequence:

$$\cdots \to KK_n^G(A,B) \xrightarrow{q\times} KK_n^G(A,B) \to KK_n^G(A,B;\mathbb{Z}_q) \to \\ \to KK_{n-1}^G(A,B) \xrightarrow{q\times} KK_{n-1}^G(A,B) \to \cdots \tag{3.2}$$

(3) there is a natural twosided exact sequence:

$$\xrightarrow{\acute{q}} KK_n^G(A,B,\mathbb{Z}_{pq}) \xrightarrow{\grave{p}} KK_n^G(A,B;\mathbb{Z}_q) \to \\ \to KK_{n-1}^G(A,B,\mathbb{Z}_p) \xrightarrow{\acute{q}} KK_{n-1}^G(A,B,\mathbb{Z}_{pq}) \to \tag{3.3}$$

(4) There is a associative product

$$KK_n^G(A,B;\mathbb{Z}_p) \otimes KK_m^G(A,B;\mathbb{Z}_q) \to KK_{n+m-2}^G(A,B;\mathbb{Z}_{pq}) \tag{3.4}$$

(5) there is a natural twosided exact sequence:

$$\cdots \to KK_n^G(A,B,\mathbb{T}) \xrightarrow{\grave{q}} KK_n^G(A,B;\mathbb{T}) \xrightarrow{\check{p}} KK_n^G(A,B,\mathbb{Z}_q) \\ \to KK_{n-1}^G(A,B,\mathbb{T}) \to \dots \tag{3.5}$$

(6) there is a natural twosided exact sequence:

$$\cdots \to KK_n^G(A,B) \xrightarrow{r} KK_n^G(A,B;\mathbb{Q}) \xrightarrow{\check{t}} KK_n^G(A,B,\mathbb{T}) \\ \to KK_{n-1}^G(A,B,) \to \dots \tag{3.6}$$

In addition there is an associative product

$$KK_n^G(A,B;\mathbb{Q}) \otimes KK_m^G(B,C;\mathbb{Q}) \to KK_{n+m}^G(B,C;\mathbb{Q})$$

Tensor product is considered over ring of integers. The product is a composition of the isomorphism:

$$(KK_n^G(A,B) \otimes \mathbb{Q}) \otimes (KK_n^G(B,C) \otimes \mathbb{Q}) \cong \\ \cong (KK_n^G(X;A,B) \otimes KK_n^G(X;B,C)) \otimes (\mathbb{Q} \otimes \mathbb{Q}), \tag{3.7}$$

which is the composition of the twisting and associativity isomorphisms of tensor product, and a homomorphism

$$(KK_n^G(A,B) \otimes KK_n^G(B,C)) \otimes (\mathbb{Q} \otimes \mathbb{Q}) \longrightarrow KK_n^G(A,C) \otimes \mathbb{Q} \tag{3.8}$$

defined by a map $(f \otimes r) \otimes (f' \otimes r') \mapsto (f \cdot f)' \otimes rr'$, where $f \cdot f$ is Kasparov product of f and f'.

Thus we can form an additive category $KK_{\mathbb{Q}}^G$, where GC^*-algebras are objects and the group of morphisms from A to B is given by the equality

$$KK_n^G(A,B;\mathbb{Q}) = KK_n^G(X;A,B) \otimes \mathbb{Q}. \tag{3.9}$$

There is a natural additive functor

$$Rat : KK^G \longrightarrow KK_{\mathbb{Q}}^G$$

which is identity on objects, and on morphisms is defined by the map $f \mapsto f \otimes 1$. It is clear that Rat is an additive functor.

The result below says that $KK^G_{\mathbb{Q}}$ is a bivariant theory on the category of separable GC^*-algebra and it is said to be the *rational* KK^G-theory.

Theorem 3.4. *The additive category $KK^G_{\mathbb{Q}}$ is a bivariant theory on the category of separable GC^*-algebras, i.e. has all fundamental properties of usual bivariant KK-theory. Besides, $KK^G_{\mathbb{T}}$ is a bimodule on the category KK^G such that it is cohomological functor relative the first argument and homological functor relative to the second argument satisfying the Bott periodicity property.*

Proof. This is easy consequence of the fact that $\mathbb{Q}$ is a flat $\mathbb{Z}$-module and the tensor product on a flat module preserves exactness. □

3.2. **A look at Baum-Connes Conjecture.** In the formulation of Baum-Connes Conjecture a crucial role play the groups $K^{top}_n(G, A)$, so called the topological K-theory of G with coefficients in A, and the homomorphism

$$\mu_A : K^{top}_n(G, A) \to K_n(G \ltimes_r A),$$

which is called the Baum-Connes assembly map. The Baum-Connes Conjecture for G with coefficients in A asserts that this map is an isomorphism. Note that $K^{top}_n(G,\ -)$ and $K_n(G \ltimes_r\ -)$ are homology theories in the sense that we have defined in the first section (cf. [10]). Therefore we have rational, torsion and finite versions of Baum-Connes Conjecture:

- (Rational version) the assembly map
$$\mu_A \otimes id_{\mathbb{Q}} : K^{top}_n(G, A) \otimes \mathbb{Q} \to K_n(G \ltimes_r A) \otimes \mathbb{Q}$$
is an isomorphism;
- (Finite version) the q-finite assembly map
$$\mu^{(q)}_A : K^{top}_n(G, A; \mathbb{Z}_q) \to K_n(G \ltimes_r A; \mathbb{Z}_q)\otimes$$
is an isomorphism;
- (Torsion version) the torsion assembly map
$$\mu^{(q)}_A : K^{top}_n(G, A; \mathbb{T})) \to K_n(G \ltimes_r A; \mathbb{T}))$$
is an isomorphism.

According to Corollary 2.2, we have the following theorem

Theorem 3.5. *The following Conjectures are equivalent.*

(1) *Baum-Connes Conjecture;*
(2) *Baum-Connes rational and torsion Conjectures;*
(3) *Baum-Connes rational and q-finite Conjectures for all primes.*

4. Remarks on finite algebraic and topological K-theories

We begin with some preliminary definitions and properties. In [2], Browder has defined algebraic K-theory of an unital ring with coefficients in $\mathbb{Z}/q$, $q \geq 2$ as follows:

$$K_n(R; \mathbb{Z}/q) = \pi_n(BGL(R)^+; \mathbb{Z}/q)$$

by using so called homotopy groups with coefficients in $\mathbb{Z}/q$.

Remark. Bellow "Algebraic K-theory of an unital ring with coefficients in $\mathbb{Z}/q$" will be replaced by "q-finite algebraic K-theory of an unital ring".

For our purposes we use equivalent definition used in [1]:

$$K^a_{n+1}(R;\mathbb{Z}/q) = \pi_n(F_q(BGL(R)^+)).$$

Here, in general, $F_q(X)$ is defined as the homotopy fiber of the q-power map of a loop space $X = \Omega Y$ (see [1]).

There exists similar interpretation for q-finite topological K-theory of C^*-al gebras. If A is an unital C^*-algebra. Then $GL(A)$ has the standard topology induced by the norm in A. Denote this topological group by $GL^t(A)$. It is known that $GL^t(A)$ and $\Omega B(GL^t(A))$ are homotopy equivalent spaces. Therefore topological K-groups may be defined equivalently by the equality

$$K^t_n(A) = \pi_n(B(GL^t(A))), \quad n \geq 1.$$

Therefore one can define the q-finite topological K-theory as follows:

$$K^t_{n+1}(R;\mathbb{Z}/q) = \pi_n(F_q(B(GL^t(R)))).$$

We have natural, up to homotopy, maps

$$B(GL(A))^+ \to B(GL^t(A))$$

and

$$F_qB(GL(A))^+ \to F_qB(GL^t(A)).$$

Therefore we have natural homomorphisms

$$\alpha_n : K^a_n(A) \to K^t_n(A) \qquad \text{and} \qquad \alpha_{n,q} : K^a_n(A,\mathbb{Z}/q) \to K^t_n(A,\mathbb{Z}/q),$$

$n \geq 1, \ \ q \geq 2$.

Proposition 4.1. *Let A be a C^*-algebra and $\mathcal{K}$ be a C^*-algebra of compact operators on a separable Hilbert space. Then the natural homomorphisms*

$$\varepsilon^{-1}\alpha_{n,q} : K^a_n(A\otimes\mathcal{K},\mathbb{Z}/q) \to K^t_n(A,\mathbb{Z}/q) \quad n \geq 1, \ \ q \geq 2,$$

are isomorphisms, where $\varepsilon : K^t_n(A;\mathbb{Z}_q) \xrightarrow{\cong} K^t_n(A\otimes\mathcal{K};\mathbb{Z}_q)$ is the isomorphism of stability for the finite topological K-theory of C^*-algebras.

Proof. It is enough to show that the homomorphism

$$\alpha_{n,q} : K^a_n(A\otimes\mathcal{K};\mathbb{Z}_q) \to K^t_n(A\otimes\mathcal{K};\mathbb{Z}_q)$$

is an isomorphism. To this end consider the following commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K^a_{n+1}(A\otimes\mathcal{K};\mathbb{Z}_q) \arrow[r] \arrow[d, "\alpha_{n+1,q}"] & K^a_n(A\otimes\mathcal{K}) \arrow[r, "\times q"] \arrow[d, "\alpha_n"] & K^a_n(A\otimes\mathcal{K}) \arrow[r] \arrow[d, "\alpha_n"] & \cdots . \\
\cdots \arrow[r] & K^t_{n+1}(A\otimes\mathcal{K};\mathbb{Z}_q) \arrow[r] & K^t_n(A\otimes\mathcal{K}) \arrow[r, "\times q"] & K^t_n(A\otimes\mathcal{K}) \arrow[r] & \cdots
\end{tikzcd}$$

Since the natural homomorphisms $\alpha_n : K^a_n(A\otimes\mathcal{K}) \to K^t_n(A\otimes\mathcal{K})$ are isomorphisms for any integer n [12], then by the Five Lemma the homomorphism

$$K^a_n(A\otimes\mathcal{K};\mathbb{Z}_q) \to K^t_n(A\otimes\mathcal{K};\mathbb{Z}_q)$$

is an isomorphism too for all $n \geq 2$.

□

5. Browder-Karoubi-lambre 's theorem for finite KK-theory

One has the following interpretation of the q-finite topological K-theory.

Proposition 5.1. *There are a natural isomorphisms*

$$K_n^t(A;\mathbb{Z}/q) \;\cong\; K_{n-2}^t(A\otimes C_q),$$

for all $n \geq 1$ and $q \geq 2$.

Proof. Since classifying space construction has functorial property, according to the functorial property of the functor $B(GL^t(-))$ and the commutative diagram

$$\begin{tikzcd} A\otimes C_q \arrow[r] \arrow[d] & A\otimes C_0(S^1)\otimes C[0;1) \arrow[d] \\ A\otimes C_0(S^1) \arrow[r] & A\otimes C_0(S^1), \end{tikzcd}$$

one gets the commutative diagrams

$$\begin{tikzcd} B(GL^t(A\otimes C_q)) \arrow[r] \arrow[d] & B(GL(A\otimes C_0(S^1)\otimes C[0;1))) \arrow[d] \\ B(GL^t(A\otimes C_0(S^1))) \arrow[r] & B(GL(A\otimes C_0(S^1))) \end{tikzcd}$$

and

$$\begin{tikzcd} F_q(\Omega B(GL^t(A))) \arrow[r] \arrow[d] & \Omega B(GL^t(A))^{[0,1)} \arrow[d] \\ \Omega B(GL^t(A) \arrow[r, "q"] & \Omega B(GL^t(A)) \end{tikzcd}$$

Since the second diagram is universal, there exists a natural map

$$\chi : B(GL^t(A\otimes C_q)) \to F_q(\Omega B(GL^t(A))).$$

Therefore one has a natural homomorphism

$$\pi_n\chi : \pi_n(B(GL^t(A\otimes C_q))) \to \pi_n(\Omega F_q(B(GL^t(A))))$$

Thus there is a natural homomorphism

$$\chi_n : K_n^t(A\otimes C_q) \to K_{n+2}^t(A,\mathbb{Z}/q).$$

Now, consider the following commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & K_n^t(A\otimes C_q) \arrow[r] \arrow[d, "\chi_n"] & K_n^t(A\otimes C_0(S^1)) \arrow[r, "\times q"] \arrow[d, "="] & K_n^t(A\otimes C_0(S^1)) \arrow[r] \arrow[d, "="] & \cdots \\ \cdots \arrow[r] & K_{n+2}^t(A,\mathbb{Z}/q) \arrow[r] & K_{n+1}^t(A) \arrow[r, "\times q"] & K_{n+1}^t(A) \arrow[r] & \cdots \end{tikzcd}$$

According to the Five Lemma, one concludes that χ_n are isomorphisms, $n \geq 1$. $\square$

Let $H : \mathcal{C}^* \to Ab$ be a functor, where $\mathcal{C}^*$ is the category of unital C^*- algebras and their homomorphisms (non-unital). Then

(1) if the inclusion in the upper left corner $A \hookrightarrow M_n(A)$ induces isomorphism $H(A) \cong H(M_n(A))$, H is said to be matrix invariant functor.

(2) if H commutes with direct system of C^*-algebras, H is said to be continuous.

For a given matrix invariant and continuous functor H there exists an extension $\mathcal{H}$ of it on the category of small additive C^*-categories $Add\ C^*$ such that the following diagram

$$\begin{tikzcd} C^* \arrow[rr, "proj_f"] \arrow[dr, "H"'] & & Add\ C^* \arrow[dl, "\mathcal{H}"] \\ & Ab & \end{tikzcd}$$

commutes, where $proj_f$ is a functor which sends unital C^*-algebra A to the additive C^*-category of finitely generated projective A-modules. The functor $\mathcal{H}$ is defined by the following manner (cf.[7], [8]).

First note that the functor H is a inner invariant functor (see Lemma 2.6.12 in [6]). Let $\mathbb{A}$ be an additive C^*-category. Set $\mathcal{L}(a) = \hom_{\mathbb{A}}(a, a)$, $a \in ob\mathbb{A}$. Let us write $a \leq a'$ if there is an isometry $v : a \to a'$ in A, i.e. $v^*v = id_a$. The relation "$a \leq a$" makes the set of objects into a directed set.

Any isometry $v : a \to a'$ in A defines a $*$-homomorphism of C^*-algebras

$$\mathrm{Ad}(v) : \mathcal{L}(a) \to \mathcal{L}(a')$$

by the rule $x \mapsto vxv^*$.

Using technics from [7], one has the following. Let $v_1 : a \to a'$ and $v_2 : a \to a'$ be two isometries in $\mathbf{A}$. Then the homomorphisms

$$\mathrm{Ad}_* v_1,\ \ \mathrm{Ad}_* v_2 : H(\mathcal{L}(a)) \to H(\mathcal{L}(a'))$$

are equal. Indeed, let $u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ be the unitary element in an unital C^*-algebra $M_2(\mathcal{L}(a'))$. Since H is a matrix invariant functor, it is *inner invariant* functor too (see Lemma 2.6.12 in [6]), i.e. the homomorphism $H(ad(u))$ is the identity map. Therefore, the maps

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad x \mapsto \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$$

sending $\mathcal{L}(a')$ into $M_2(\mathcal{L}_A(I)(a'))$, induces the same isomorphisms after applying the functor H. It is clear that the homomorphism $\nu_*^{aa'} = H(\nu^{aa'})$ is not depending on the choice of an isometry $\nu^{aa'} : a \to a'$. Therefore one has a direct system $\{H(\mathcal{L}(a)), \nu_*^{aa'})\}_{a,a' \in obA}$ of abelian groups.

Definition 5.2. Let $\mathbb{A}$ be an additive small C^*-category. Then by definition

$$\mathcal{H}(\mathbb{A}) = \varinjlim H(\mathcal{L}(a)).$$

So defined functor makes commutative the above diagram. That follows from the matrix invariant and continuous properties of H and is a simple exercise (see [8]).

Since the functors $K^t(-; \mathbb{Z}/q)$ have the above mentioned properties, one can define the q-finite topological K-theory for an additive C^*-category $\mathbb{A}$ by setting

$$K_n^t(\mathbb{A}; \mathbb{Z}/q) = \varinjlim K_n^t(\mathcal{L}(a); \mathbb{Z}/q).$$

This definition is in accordance with other definitions of q-finite topological K-theories because of the matrix invariant and continuous properties. Therefore we

get a generalization of Browder-karoubi-Lambre 's theorem for small additive C^*-categories.

Proposition 5.3. *Let $\mathbb{A}$ be a small additive C^*-category. Then, for all $n \in \mathbb{Z}$,*

(1) $q \cdot K^t_n(\mathcal{A}, Z/q) = 0$, *if $q - 2$ is not divided by 4;*
(2) $2q \cdot K^t_n(\mathcal{A}, Z/q) = 0$, *if 4 divides $q - 2$.*

Proof. It is consequence of Proposition 4.1. □

The next step is to give an interpretation of q-finite KK^G-theory as topological K-theory of the additive C^*-category $Rep_G(A, B)$. Such an interpretation exists for KK^G-theory, where G is a compact metrizable group [8].

Theorem 5.4. *Let A and B be, respectively, separable and σ-unital $G - C^*$-algebras, real or complex; and G be metrizable compact group. Then, for all integer n and $q \geq 2$, there exists a natural isomorphisms*

$$KK^G_n(A, B; \mathbb{Z}_q) \cong K^t_{n+1}(\mathrm{Rep}(A, B); \mathbb{Z}/q),$$

When G is locally compact group, the proof is more complicated and this case will be investigated in the further paper.

First we recall the definition of the C^*-category $Rep(A, B)$. This category was constructed in [8].

Let $\mathcal{H}_G(B)$ be the additive C^*-category of countably generated right Hilbert B-modules equipped with a B-linear, norm-continuous G-action over a fixed compact second countable group G [9]. Note that the compact group acts on the morphisms by the following rule: for $f : E \to E'$ the morphism $gf : E \to E'$ is defined by the formula $(gf)(x) = g(f(g^{-1}(x)))$.

The category $\mathcal{H}_G(B)$ contains the class of compact B-homomorphisms [9]. Denote it by $\mathcal{K}_G(B)$. Known properties of compact B-homomorphisms imply that $\mathcal{K}_G(B)$ is a C^*-ideal [4] in $\mathcal{H}_G(B)$.

Objects of the category $Rep(A, B)$ are pairs of the form (E, φ), where E is an object in $\mathcal{H}_G(B)$ and $\varphi : A \to \mathcal{L}(E)$ is an equivariant $*$-homomorphism. A morphism $f : (E, \phi) \to (E', \phi')$ is a G-invariant morphism $f : E \to E'$ in $\mathcal{H}_G(B)$ such that

$$f\phi(a) - \phi'(a)f \in \mathcal{K}_G(E, E')$$

for all $a \in A$. The structure of a C^*-category is inherited from $\mathcal{H}_G(B)$. It is easy to see that $Rep(A, B)$ is an additive C^*-category, not idempotent-complete.

Now, we are ready to construct our main C^*-category, that is $\mathrm{Rep}(A, B)$. Its objects are triples (E, ϕ, p), where (E, ϕ) is an object and $p : (E, \phi) \to (E, \phi)$ is a morphism in $Rep(A, B)$ such that $p^* = p$ and $p^2 = p$. A morphism $f : (E, \phi, p) \to (E', \phi', p')$ is a morphism $f : (E, \phi) \to (E', \phi')$ in $Rep(A, B)$ such that $fp = p'f = f$. In detail, f must satisfy

$$f\phi(a) - \phi'(a)f \in \mathcal{K}(E, F) \text{ and } fp = p'f = f. \tag{5.1}$$

So, by definition

$$\mathrm{Rep}(A, B) = \widetilde{Rep(A, B)}.$$

The structure of a C^*-category on $\mathrm{Rep}(A, B)$ comes from the corresponding structure on $Rep(A, B)$.

Proof. (*of the theorem* 5.5) The following isomorphisms

$$\theta^a_n : K^a_n(\mathrm{Rep}(A;B)) \simeq KK^G_{n-1}(A;B),$$

and

$$\theta^t_n : K^t_n(\mathrm{Rep}(A;B)) \simeq KK^G_{n-1}(A;B),$$

was proved in [8]. According to the definition of the finite KK^G-groups and these isomorphisms, in particular, we have the following result for finite KK^G-theory:

Let A and B be, respectively, separable and σ-unital $G-C^*$-algebras. Then

$$KK^G_n(A,B;\mathbb{Z}_q) \cong K^t_{n-1}(\mathrm{Rep}(A;B\otimes C_q)) \cong K^a_{n-1}(\mathrm{Rep}(A;B\otimes C_q)). \tag{5.2}$$

Therefore it is enough to show that

$$K^t_{n+1}(\mathrm{Rep}(A,B);\mathbb{Z}/q) \cong K^t_{n-1}(\mathrm{Rep}(A;B\otimes C_q)).$$

Note that

$$K^t_{n-1}(\mathrm{Rep}(A,B\otimes C_q) \cong \varinjlim\nolimits_{a\in \mathrm{Rep}(A,B\otimes C_q)} \ K^t_{n-1}(\mathcal{L}(a))$$

and

$$K^t_{n+1}(\mathrm{Rep}(A,B;Z_q)) = \varinjlim\nolimits_{b\in ob\mathrm{Rep}(A,B)} \ K^t_{n-1}(\mathcal{L}(b)\otimes C_q).$$

So it is enough to compare the right-hand sides.

Consider $\mathrm{Rep}(A,B)\otimes C_q$ as the C^*-tensor product of C^*-categoroids in the sense of [8] (or as non-unital C^*-categories in the sense of [11]).

There is a natural (non-unital) functor

$$\nu : Rep(A,B)\otimes C_q \to \mathrm{Rep}(A,B\otimes C_q)$$

defined by maps:

(1) $b = (\varphi, E, p) \mapsto \varphi\otimes id_{C_q}, E\otimes C_q, p\otimes \mathrm{id}_{C_q}) = a_b$ on objects;
(2) $f \mapsto f\otimes id_{C_q}$ on morphisms.

One has induced morphism of direct systems of abelian groups

$$\{\nu_a\} : \{K^t_n(\mathcal{L}(a)\otimes C_q)\} \to \{K^t_n(\mathcal{L}(b)\},$$

where $\nu_a : K^t_n(\mathcal{L}(a)\otimes C_q) \to K^t_n(\mathcal{L}(a_b)$ is induced by ν. Therefore one has a natural homomorphism

$$\bar\nu_n : K^t_{n+1}(\mathrm{Rep}(A,B);\mathbb{Z}/q) \to K^t_{n-1}(\mathrm{Rep}(A;B\otimes C_q)).$$

Then comparing the two twosided exact sequences

$$\begin{tikzcd}
\cdots \arrow[r] & K^t_{n+1}(\mathrm{Rep}(A;B),\mathbb{Z}_q) \arrow[r] \arrow[d, "\bar\nu_n"] & K^t_n(\mathrm{Rep}(A;B)) \arrow[r, "\times q"] \arrow[d, "="] & K^t_n(\mathrm{Rep}(A;B)) \arrow[r] \arrow[d, "="] & \cdots \\
\cdots \arrow[r] & K^t_{n-1}(\mathrm{Rep}(A;B\otimes C_q)) \arrow[r] & K^t_n(\mathrm{Rep}(A;B)) \arrow[r, "\times q"] & K^t_n(\mathrm{Rep}(A;B)) \arrow[r] & \cdots
\end{tikzcd}$$

one concludes that $\bar\nu$ is an isomorphism. □

Now,we show the Browder-Karoubi-Lambre's theorem for finite KK^G-theory .

Theorem 5.5. *Let A and B be, respectively, separable and σ-unital $G-C^*$-algebras, real or complex; and G be metrizable compact group. Then, for all $n\in\mathbb{Z}$,*

(1) $q\cdot KK^G_n(A,B;\mathbb{Z}_q) = 0,$ *if $q-2$ is not divided by 4;*
(2) $2q\cdot KK^G_n(A,B;\mathbb{Z}_q) = 0,$ *if 4 divides $q-2$.*

Proof. Follows from Propositions 4.1, 5.1 and 5.3, from Theorem 5.4 and from the Browder-Karoubi-Lambre's theorem for algebraic K-theory. □

References

[1] Arlettaz D. Inassaridze Finite K-theory spaces Math. Proc. Camb. Phil.Soc. (2005), 139. 261-286.

[2] Browder W. Algebraic K-theory with coefficients $\mathbb{Z}/p$, in Geometric Applications of Homotopy Theory I, Lect. Notes in Math. 657 (Springer, 1978), 40-84.

[3] Blackadar B. *K-theory for Operator Algebras*, M.S.R.I. Publ. 5, Springer-Verlag, (1986).

[4] P. Chez, R. Lima and J. Roberts, *W^*-categories*, Pacific J. Math. **120** (1985), No 1, 79-109.

[5] Cuntz J., Meyer R., Rosenberg J., *Topological and Bivariant K-Theory*, Birkhuser, Basel (2007).

[6] Higson N. *Algebraic K-theory of stable C^*-algebras*, Adv. Math., v.67, (1988) 1-140.

[7] Higson N. *C^*-algebra extension theory and duality* , J.Funct. Anal., v.129, (1995), 349-363.

[8] Kandelaki T., *Algebraic K-theory of Fredholm modules and KK-theory* Vol. 1, (2006), No. 1, 195-218.

[9] Kasparov G. *Hilbert C^*-modules: Theorems of stinespring and voiculescu*, J. Operator theory, v. 4, (1980), 133-150.

[10] Meyer R., Nest R. The Baum-Connes Conjecture via Localisation of Categories arXiv:math/0312292 .

[11] Mitchener P. C*-*categories* Proceedings of the London Mathematical Society, volume 84 (2002), 375-404.

[12] Suslin A., Wodzicki M. *Excision in algebraic K-theory*, Ann. Math. v.136, No.1, (1992), 51-122.

[13] Weidner J. *Topological invariant for Generalized Operatorial Algebras* Dissertation, Heidelberg (1987).

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